Symmetry Properties of Higher Numerical Ranges

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ABSTRACT

Let A be a linear operator on a finite dimensional unitary space V of dimension n. The kth higher numerical range of A, denoted by $W_k(A)$, is the totality of complex numbers $\operatorname{tr}(PAP)$ where P runs over all k-dimensional orthogonal projections on V. We show that $W_k(A)$ is a polygon with the real axis as a line of symmetry, $k=1,\ldots,n$, if and only if A is normal with a real characteristic polynomial. We also construct several nonnormal examples to investigate the extent to which the symmetry of all of the $W_k(A)$ is required.

I. INTRODUCTION

Let V be an n-dimensional unitary space with inner product (x,y). If $A:V\to V$ is linear, then the image of the surface of the unit sphere under the mapping

$$x \to (Ax, x) \tag{1}$$

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is called the *numerical range* of A. Once an orthonormal basis of V is chosen, A may be regarded as an n-square complex matrix, and the numerical range of A becomes

$$W(A) = \{x^*Ax | x^*x = 1\}, \tag{2}$$

where x is a complex $n \times 1$ matrix.

There is a rich literature on $W(A)$ that dates back to the last century. To quote P. R. Halmos [2],

> In early studies of Hilbert space (by Hilbert, Hellinger, Toeplitz, and others) the objects of chief interest were quadratic forms.

Certainly the most interesting and important theorem about $W(A)$ is the Toeplitz-Hausdorff theorem [3, 9] to the effect that $W(A)$ is a convex set in the plane. This theorem was the first in a long series of results that concern the interplay between the algebraic properties of the operator A and the geometric properties of $W(A)$ and its numerous generalizations. The importance of the Toeplitz-Hausdorff theorem lies in its universal applicability—there are no hypotheses on A.

There are a large number of relations between algebraic operations on A and geometric properties of $W(A)$. We shall repeatedly use the following facts:

$$W(cA) = cW(A); \tag{3}$$

$$W(cI_n + A) = c + W(A); \tag{4}$$

$$W(U^*AU) = W(A), \qquad U \text{ unitary}; \tag{5}$$

$$W(A^*) = \overline{W(A)}, \qquad W(A^T) = W(A); \tag{6}$$

$$W(A \oplus B) = H(W(A) \cup W(B)), \qquad H \text{ denotes convex hull}; \tag{7}$$

$$W(B) \subset W(A), \qquad B \text{ a principal submatrix of } A. \tag{8}$$

The numerical range of a 2-square matrix was explicitly computed by F. D. Murnaghan [8].

Elliptical range theorem. *Let A be a 2-square complex matrix with eigenvalues λ and μ. Let $\|A\|$ denote the Euclidean norm, and define $\alpha \geq 0$ by the equation*

$$\alpha^2 = \|A\|^2 - |\lambda|^2 - |\mu|^2. \tag{9}$$

Then $W(A)$ is an elliptical region with foci λ and μ, major axis of length

$$\left(\|A\|^2 - 2\operatorname{Re}\lambda\bar{\mu}\right)^{1/2}, \tag{10}$$

and minor axis of length

$$\alpha. \tag{11}$$

Moreover, A is unitarily similar to the upper triangular matrix

$$\begin{bmatrix} \lambda & \alpha \\ 0 & \mu \end{bmatrix}. \tag{12}$$

In a recent paper one of the present authors and C. Pesce proved that the numerical range of any complex matrix A can be obtained as the union of numerical ranges of all two dimensional real compressions of A [6]. More precisely,

$$W(A) = \bigcup W(A_{xv}), \tag{13}$$

where

$$A_{xv} = \begin{bmatrix} (Ax,x) & (Av,x) \\ (Ax,v) & (Av,v) \end{bmatrix} \tag{14}$$

and x and v run over all *real* orthonormal pairs of vectors. This result made possible the construction of a program to exhibit $W(A)$ graphically for matrices of modest dimension.

Thirty-two years ago one of the present authors and B. N. Moyls [4] proved that for $n \leqslant 4$, $W(A)$ is a polygonal region in the plane if and only if A is normal. It is easy to construct 5-square nonnormal matrices for which $W(A)$ is a polygonal region.

In his thesis, C. A. Berger [1] proved that the higher numerical range of an operator is convex. Actually, Berger's result is subsumed by a theorem of Westwick [10] to the effect that

$$W_c(A) = \left\{ \sum_{j=1}^{n} c_j(Ax_j, x_j) \,\middle|\, x_1, \ldots, x_n \text{ orthonormal} \right\} \tag{15}$$

is convex for any real $c = [c_1, \dots, c_n]$. The set $W_c(A)$ is called the *c-numerical range*, and if $c_1 = \cdots = c_k = 1$, $c_{k+1} = \cdots = c_n = 0$, then $W_c(A)$ is denoted by $W_k(A)$ and is called the kth *higher numerical range* [2]. In fact,

$$W_k(A) = \{\mathrm{tr}(PAP) | P \text{ a } k\text{-dimensional orthogonal projection}\}. \quad (16)$$

We use the symbols $W(A)$ and $W_1(A)$ interchangeably.

In case A is normal, more can be said about $W_c(A)$. Let $P_c(A)$ denote the convex polygon spanned by the points

$$\sum_{j=1}^{n} c_j \lambda_{\sigma(j)}, \quad (17)$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, and σ runs over the symmetric group of degree n [5]. The principal result in [5] provides necessary and sufficient conditions for A to be normal in terms of the structure of each $W_k(A)$. To be precise, in the case of $W_k(A)$, let the convex polygon spanned by (17) ($c_1 = \cdots = c_k = 1$, $c_{k+1} = \cdots = c_n = 0$) be denoted by $P_k(A)$. Then

$$W_k(A) = P_k(A), \qquad k = 1, \dots, n, \quad (18)$$

if and only if A is normal.

Since

$$W_{n-k}(A) = \mathrm{tr}(A) - W_k(A), \quad (19)$$

the equality (18) need only be assumed for $k = 1, \dots, [n/2]$ in order to conclude that A is normal.

In [7] we obtained necessary and sufficient conditions for $W_c(A) \subset \mathbb{R}$. Without going into detail, the results show how A is related to a Hermitian matrix.

In the present paper we seek necessary and sufficient conditions on a normal matrix A so that the higher numerical ranges are symmetric across the x-axis, i.e.,

$$\overline{W_k(A)} = W_k(A). \quad (20)$$

We note before going on that (3), (5), and (6) are true when W is replaced by W_k. Also, (4) becomes

$$W_k(cI_n + A) = ck + W_k(A). \quad (21)$$

Let S be a subset of the plane. Then S is said to be *x-symmetric* if for any z in S, the complex conjugate $\bar{z}$ is also in S. Thus, an x-symmetric set S must satisfy

$$\bar{S} = S. \tag{22}$$

Our principal results follow.

THEOREM 1. *Let A be an n-square complex normal matrix. Then $W_k(A)$ is x-symmetric for $k = 1,\ldots,n$ if and only if the characteristic polynomial of A has real coefficients.*

COROLLARY 1. *Let A be an n-square complex normal matrix. Then A is unitarily similar to a real normal matrix if and only if $W_k(A)$ is x-symmetric for $k = 1,\ldots,[n/2]$, and* $\mathrm{tr}(A)$ *is real.*

We are indebted to the referee for pointing out that if the hypotheses in Corollary 1 are changed so that $W_k(A)$ is assumed to be x-symmetric for $k = 1,\ldots,[(n+1)/2]$, then the additional condition that $\mathrm{tr}(A)$ is real is unnecessary. We also wish to point out that Equation (19) actually shows that our formulation and that of the referee are equivalent. If n is even, $n = 2m$, then $m = [(n+1)/2]$, so that following the referee, $W_m(A)$ is x-symmetric. From (19), $W_m(A)$ is a translate by $\mathrm{tr}(A)$ of the x-symmetric set $-W_m(A)$. But then it follows that $\mathrm{tr}(A)$ must be real. Thus when $n = 2m$, the referee's suggestion shows that the reality of $\mathrm{tr}(A)$ is a consequence of the x-symmetry of $W_m(A)$. If $n = 2m+1$ then $[(n+1)/2] = m+1$. If $W_m(A)$ and $W_{m+1}(A)$ are x-symmetric, then by (19) and the preceding argument we again conclude that $\mathrm{tr}(A)$ is real. Thus, in any case the referee's suggestion implies that $\mathrm{tr}(A)$ is real. Conversely, (19) and the hypotheses in Corollary 1 immediately imply that $W_k(A)$ is x-symmetric for $k = [(n+1)/2]$.

For a nonnormal matrix A, it is not necessarily true that $W_k(A)$ is a polygon. However, we can state the following result.

THEOREM 2. *Let A be an n-square complex matrix. Then A is unitarily similar to a real normal matrix if and only if $W_k(A)$ is an x-symmetric polygon for $k = 1,\ldots,[n/2]$ and* $\mathrm{tr}(A)$ *is real.*

Before we present proofs of these theorems we will analyze several examples that exhibit precisely the extent to which the hypotheses are required.

II. EXAMPLES

For $n = 2$ the situation is unique.

THEOREM 3. *If A is a 2-square complex matrix, then $W(A)$ is x-symmetric if and only if A is unitarily similar to a real matrix.*

Proof. Assume that $W(A)$ is x-symmetric. By the elliptical range theorem, $W(A)$ is an ellipse (possibly degenerate). Hence either the major or the minor axis must lie along the x-axis. [If $W(A)$ is a circular disk, then the foci are collapsed to the center and there is a diameter along the x-axis.] It follows that the eigenvalues λ and μ of A are either both real or are complex conjugates.

Case 1. λ *and* μ *are real.* By the elliptical range theorem [see (12)] A is unitarily similar to the matrix

$$\begin{bmatrix} \lambda & \alpha \\ 0 & \mu \end{bmatrix}. \tag{23}$$

The matrix (23) is similar via a diagonal unitary similarity to a matrix identical to (23) with $\alpha \geqslant 0$.

Case 2. $\lambda = r + is$, $\mu = r - is$, $s \neq 0$. Again, we may assume A is the matrix (23):

$$\begin{bmatrix} r+is & \alpha \\ 0 & r-is \end{bmatrix} = rI_2 + s\begin{bmatrix} i & \alpha/s \\ 0 & -i \end{bmatrix},$$

where r and s are real and $\alpha \geqslant 0$. By (3) and (4), $W(A)$ is x-symmetric if and only if the numerical range of

$$B = \begin{bmatrix} i & \alpha \\ 0 & -i \end{bmatrix} \tag{24}$$

is x-symmetric, where we have replaced α/s in (23) by α for notational simplicity. Define a real matrix

$$R = \begin{bmatrix} \dfrac{\alpha}{2} & \dfrac{-(4+\alpha^2)^{1/2}}{2} \\ \dfrac{(4+\alpha^2)^{1/2}}{2} & -\dfrac{\alpha}{2} \end{bmatrix}. \tag{25}$$

The classical Schur triangularization theorem states that any matrix is unitarily similar to an upper triangular matrix, known as its Schur form. The remainder of the proof consists of showing that the Schur form of R is B. The characteristic polynomial of R is λ^2+1, so that the eigenvalues of R are $\pm i$. Hence the Schur form of R can be taken to be

$$C=\begin{bmatrix} i & \rho \\ 0 & -i \end{bmatrix}, \qquad \rho \geqslant 0,$$

where by (9)

$$\rho^2 = \|C\|^2 - 2. \tag{26}$$

But

$$\begin{aligned}\|C\|^2 &= \|R\|^2 \\ &= \alpha^2+2. \end{aligned}\tag{27}$$

From (26) and (27), $\rho^2=\alpha^2$ and hence $\rho=\alpha$. Thus $C=B$. The converse is true for general n by (6). For, if A is real, so that $A^*=A^T$, then

$$\begin{aligned} W(A) &= W(A^T) \\ &= W(A^*) \\ &= \overline{W(A)} \end{aligned}$$

and hence $W(A)$ is x-symmetric. ∎

Theorem 3 suggests the possibility that the x-symmetry of $W(A)$ implies that A is unitarily similar to a real matrix for $n>2$. However, the following example shows that this is not true.

Example 1. Let

$$A=\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & z \end{bmatrix}, \tag{28}$$

where z satisfies

$$0<|z|<1 \tag{29}$$

and

$$\theta = \arg z \neq k\pi/2 \tag{30}$$

for any integer k. We show that $W(A)$ is x-symmetric but that no nonzero complex multiple of A is unitarily similar to a real matrix. The numerical range of the upper left 2-square principal submatrix of A is an ellipse with foci at 1 and -1 and minor axis of length 2. The condition (29) implies that z is in the interior of this ellipse, and hence by (7) the numerical range of A is the indicated ellipse. Assume that

$$R = e^{i\varphi}A \tag{31}$$

is unitarily similar to a real matrix. Then the eigenvalues of R are all real, or are two complex conjugates and a real. The eigenvalues of R are

$$e^{i\varphi}, \qquad -e^{i\varphi}, \qquad e^{i\varphi}z. \tag{32}$$

Case 1. The eigenvalues of R are all real. From (32), φ must be an integral multiple of π, and hence

$$e^{i\varphi}z = e^{i(\varphi+\theta)}|z|, \tag{33}$$

where $\varphi + \theta$ must be an integral multiple of π. It follows that θ is an integral multiple of π, contradicting (30).

Case 2. The eigenvalues of R are two complex conjugates and a real. Since $|e^{i\varphi}z| < 1$ by (29), it follows that the two complex conjugate eigenvalues of (31) are $e^{i\varphi}$ and $-e^{i\varphi}$. As in case 1, we have that $\varphi + \theta$ is an integral multiple of π, i.e., $\varphi + \theta = m\pi$. Then

$$-e^{i\varphi} = e^{i(\varphi+\pi)}, \tag{34}$$

and since $e^{i\varphi}$ and $-e^{i\varphi}$ are complex conjugates, by (34) we have

$$\begin{aligned}\overline{e^{i\varphi}} &= -e^{i\varphi}\\ &= e^{i(\varphi+\pi)},\end{aligned}$$

$$e^{i(2\varphi+\pi)} = 1.$$

Thus $2\varphi + \pi$ is an integral multiple of 2π, i.e.,

$$2\varphi + \pi = 2k\pi. \tag{35}$$

Since $\varphi + \theta = m\pi$, (35) implies that

$$\theta = \{2(m-k)+1\}\frac{\pi}{2},$$

contradicting (30).

Note that Example 1 can be extended to an example of an n-square matrix, $n > 3$, for which $W(A)$ is x-symmetric but for which no nonzero multiple of A is unitarily similar to a real matrix. Simply replace z by zI_{n-2}.

The second higher numerical range of A, $W_2(A)$, consists of all sums of the form

$$w = (Ax_1, x_1) + (Ax_2, x_2) \tag{36}$$

where x_1 and x_2 are orthonormal. For a given such pair x_1 and x_2, let x_3 complete x_1 and x_2 to an orthonormal basis. Then

$$\begin{aligned} w + (Ax_3, x_3) &= \operatorname{tr}(A) \\ &= z. \end{aligned} \tag{37}$$

Hence

$$W_2(A) = z - W_1(A), \tag{38}$$

and it follows that $W_2(A)$ is the ellipse obtained by translating the ellipse $-W_1(A)$ through z. But $W_1(A) = -W_1(A)$, so that $W_2(A)$ is just $W_1(A)$ translated through z. Note that $W_2(e^{i\varphi}A) = e^{i\varphi}W_2(A)$, and thus the foci of $W_2(e^{i\varphi}A)$ are obtained from the foci $z \pm 1$ of $W_2(A)$ by multiplying by $e^{i\varphi}$:

$$e^{i\varphi}(z+1), \qquad e^{i\varphi}(z-1). \tag{39}$$

In order for $W_1(e^{i\varphi}A)$ to be x-symmetric the two foci must either both be real or be complex conjugates. Moreover, the center of the ellipse $W_2(e^{i\varphi}A)$ must be a real number. The center is the point $e^{i\varphi}z$. So suppose

$$e^{i\varphi}z = r, \tag{40}$$

so that

$$\varphi = k\pi - \theta. \tag{41}$$

The foci (39) become

$$\begin{aligned} e^{i\varphi}z \pm e^{i\varphi} &= r \pm e^{i\varphi} \\ &= r \pm e^{i(k\pi-\theta)} \\ &= r \pm e^{-i\theta}. \end{aligned} \tag{42}$$

The numbers (42) are real or complex conjugates if and only if θ is an integral multiple of $\pi/2$, contradicting (30). Thus no nonzero multiple of A has a second higher numerical range which is x-symmetric. The matrix A is clearly nonnormal, $W_1(A)$ is x-symmetric, $W_2(A)$ is not x-symmetric, and the trace is not real. If A were normal with $W_1(A)$ x-symmetric and $\operatorname{tr}(A)$ real, Corollary 1 would imply that A is unitarily similar to a real matrix. But more can be said about this example. It is generally true for any n-square complex matrix A and any k, $1 \leqslant k \leqslant n$, that if $W_k(A)$ and $W_{n-k}(A)$ are both x-symmetric then $\operatorname{tr}(A)$ must be real. Assuming for a moment that this is true, the preceding example actually exhibits a nonnormal A for which $W_1(A)$ is x-symmetric. The fact that $W_2(A)$ is not x-symmetric is then equivalent to the fact that $\operatorname{tr}(A)$ is not real. To complete this argument note that from (19), if $W_k(A)$ is x-symmetric, then $-W_k(A)$ is obviously x-symmetric, and $W_{n-k}(A)$ is thereby simply a translate of an x-symmetric set through the complex number $\operatorname{tr}(A)$. It follows that $W_{n-k}(A)$ cannot be x-symmetric unless $\operatorname{tr}(A)$ is real, i.e., a translate of any x-symmetric set is x-symmetric if and only if the translate is a real one.

Of course, it is easy to exhibit a normal 3-square matrix A for which $W_1(A)$ is x-symmetric and yet no nonzero complex multiple of A is unitarily similar to a real matrix:

$$A = \operatorname{diag}\left(1+i, 1-i, 1+\frac{i}{2}\right). \tag{43}$$

If $e^{i\varphi}A$ were unitarily similar to a real matrix, it would follow that

$$e^{i\varphi}(1+i), \qquad e^{i\varphi}(1-i), \qquad e^{i\varphi}\left(1+\frac{i}{2}\right) \tag{44}$$

consists of three real numbers or one real number and two complex conjugates. It is straightforward to confirm that no such φ can exist. However,

$$i(A-I_3)=\operatorname{diag}(-1,1,-\tfrac{1}{2})$$

is a real matrix. Thus a simple translation followed by a rotation of A will result in a real matrix. This situation is, in fact, covered by Corollary 2.

COROLLARY 2. *If A is a 3-square normal matrix and $W_1(A)$ is x-symmetric, then to within a real translation, followed by a rotation through an integral multiple of $\pi/2$, A is unitarily similar to a real matrix.*

Proof. By (18), $W_1(A)$ is a point, a line segment, or a triangle. If $W_1(A)$ is a point, it must be real, so that A is a real multiple of I_3. If $W_1(A)$ is a line segment, then either $W_1(A)\subseteq\mathbb{R}$ or $W_1(i(A-rI_3))\subseteq\mathbb{R}$, $r\in\mathbb{R}$. In either case the result follows, since either A, in the first case, or $i(A-rI_3)$, in the second case, has real eigenvalues. Finally, if $W_1(A)$ is a triangle, then the eigenvalues of A must be of the form

$$p+ir,\qquad p-ir,\qquad s,$$

where p, r, and s are real. But then A is unitarily similar to

$$\begin{bmatrix} p & r & 0\\ -r & p & 0\\ 0 & 0 & s\end{bmatrix}. \qquad \blacksquare$$

EXAMPLE 2. Let

$$A=\operatorname{diag}(i,-i,1,z), \tag{45}$$

where z satisfies

$$0<|z|<\frac{1}{\sqrt{2}}, \tag{46}$$

and if $\arg z$ is denoted by θ, then

$$0<|\theta|<\pi/2. \tag{47}$$

The matrix A is obviously normal, and (46) implies that $W_1(A)$ is the x-symmetric triangle with vertices i, $-i, 1$. We show, however, that even to within a translation and rotation, A is not unitarily similar to a real matrix. Let

$$B = e^{i\varphi}(A - pI_4), \tag{48}$$

where p is real. The eigenvalues of B are

$$e^{i\varphi}(i-p), \qquad e^{i\varphi}(-i-p), \qquad e^{i\varphi}(1-p), \qquad e^{i\varphi}(z-p). \tag{49}$$

We show that the numbers (49) cannot be roots of a real quartic. First we prove that

$$p \neq 0 \tag{50}$$

and that $e^{i\varphi}(1-p)$ cannot be one of a conjugate pair with either $e^{i\varphi}(i-p)$ or $e^{i\varphi}(-i-p)$. Assume $p = 0$. Then the numbers (49) become

$$ie^{i\varphi}, \qquad -ie^{i\varphi}, \qquad e^{i\varphi}, \qquad ze^{i\varphi}. \tag{51}$$

The numbers (51) are not all real. Moreover, they cannot be two complex conjugate pairs, because $|z| < 1$. The only alternative is that (51) consists of two reals and a complex conjugate pair. Again, $ze^{i\varphi}$ cannot be one of a complex conjugate pair, so that it must be real. But

$$ze^{i\varphi} = |z|e^{i(\theta+\varphi)},$$

and hence

$$\theta + \varphi = k\pi \tag{52}$$

for some integer k. The sum of the numbers in (51) is the real number

$$e^{i\varphi} + ze^{i\varphi},$$

and hence $e^{i\varphi}$ must be real. It follows that φ is an integral multiple of π, in contradiction to (47). Thus $p \neq 0$. Next, if $e^{i\varphi}(1-p)$ were one of a conjugate pair with either $e^{i\varphi}(i-p)$ or $e^{i\varphi}(-i-p)$, then

$$|1-p| = |\pm i - p|,$$

which implies that $p = 0$, and we saw this is not true.

Case 1. The numbers (49) *consist of two sets of conjugate pairs.* We just saw that $e^{i\varphi}(1-p)$ cannot be one of a conjugate pair with either of $e^{i\varphi}(\pm i-p)$. Hence the conjugate pairs must be

$$e^{i\varphi}(i-p), \qquad e^{i\varphi}(-i-p) \tag{53}$$

and

$$e^{i\varphi}(1-p), \qquad e^{i\varphi}(z-p). \tag{54}$$

Since the numbers (53) are complex conjugates, their sum

$$-2pe^{i\varphi} \tag{55}$$

is real, and hence $p=0$ or $e^{i\varphi}$ is real. Since $p\neq 0$, $e^{i\varphi}$ must be real, and the numbers (54) are not complex conjugates.

Case 2. The numbers (49) *consist of one complex conjugate pair and two real numbers.* Suppose first that the numbers (53) are the conjugate pair and the numbers (54) are real. Then the sum in (53), $-2pe^{i\varphi}$, is real, and since $p\neq 0$, it follows that $e^{i\varphi}$ is real. But then the last number in (49) is real, so that $z-p$ is real, and finally z is real, contradicting (47).

Next assume that (54) are the conjugate pair and (53) are the reals. If we add the numbers (53), the sum is again (55), and since $p\neq 0$, $e^{i\varphi}$ is real, and hence $e^{i\varphi}(1-p)$ is real and we conclude that (54) cannot be the conjugate pair. Since we showed above that $e^{i\varphi}(1-p)$ cannot be one of a conjugate pair with either of the numbers (53), it follows that in case 2 the only possibilities remaining are that $e^{i\varphi}(z-p)$ is one of a conjugate pair with one of the numbers (53) and that $e^{i\varphi}(1-p)$ and the other number in (53) are real. If $e^{i\varphi}$ were real, then one of the numbers (53), which are now real multiples of $i-p$ and $-i-p$, would have to be real, a contradiction. Hence $p=1$, and the numbers (49) are

$$e^{i\varphi}(i-1), \qquad e^{i\varphi}(-i-1), \qquad 0, \qquad e^{i\varphi}(z-1), \tag{56}$$

and there are a conjugate pair and two reals in (56). We have already proved at the beginning of case 2 that the first two numbers in (56) cannot be a conjugate pair. Thus the conjugate pair in the list (56) must be either

$$e^{i\varphi}(i-1), \qquad e^{i\varphi}(z-1)$$

or

$$e^{i\varphi}(-i-1), \qquad e^{i\varphi}(z-1).$$

In either case,

$$|z-1|=\sqrt{2},$$

so that z is on a circle C of radius $\sqrt{2}$ centered at 1. The point z is in the interior of $W_1(A)$, a triangle with vertices i, $-i, 1$. But $z \in C \cap W_1(A) = \{i, -i\}$, and by (47), z is neither i nor $-i$.

Case 3. The numbers (49) *are all real.* In particular, $e^{i\varphi}(1-p)$ is real, so that $p=1$ or $e^{i\varphi}$ is real. If $e^{i\varphi}$ is real, then $i-p$ is real, a contradiction. Thus $p=1$, and the sum of $e^{i\varphi}(i-p)$ and $e^{i\varphi}(-i-p)$ is $-2e^{i\varphi}$, a real number, and again we conclude that $e^{i\varphi}$ is real, with the result that $i-p=i-1$ is real.

If we replace z by zI_{n-2} in the definition (45), we have an n-square normal matrix with an x-symmetric numerical range that even to within translation and rotation is not unitarily similar to a real matrix, in distinction to the $n=2$ and $n=3$ cases.

Some additional observations can be made about the x-symmetry of the higher order numerical ranges of the matrix A in (45) of Example 2. From (18) we know that $W_2(A)$ is the convex hull of the six points

$$0, \quad 1+i, \quad 1-i, \quad z+i, \quad z-i, \quad z+1. \tag{57}$$

The real parts of the first five points in (57) are

$$0, \qquad 1, \qquad \operatorname{Re} z,$$

and hence by (46) no convex combination of the first five numbers in (57) can have a real part exceeding 1. But by (46) and (47)

$$\operatorname{Re}(z+1)=1+\operatorname{Re} z>1.$$

Thus $z+1$ is a vertex of $W_2(A)$. But $\bar{z}+1$ is not in the convex hull of the numbers (57). Hence $W_2(A)$ is not x-symmetric. However, note that $\operatorname{tr}(A)$ is not real, so that it is interesting to ask whether a 4-square normal A can exist for which $W_1(A)$ is x-symmetric, $W_4(A)=\operatorname{tr}(A)$ is real, and yet A is not unitarily similar to a real matrix. The answer is no. In other words we have the following:

THEOREM 4. *If A is a 4-square complex normal matrix, $W_1(A)$ is x-symmetric, and* $\operatorname{tr}(A)$ *is real, then A is unitarily similar to a real matrix.*

Proof. First consider the case in which $W_1(A)$ is a nondegenerate x-symmetric quadrilateral. Then the vertices, i.e., the eigenvalues of A, must pair off into complex conjugate pairs $\lambda, \bar{\lambda}$, and $\mu, \bar{\mu}$. Then A is unitarily similar to the 4-square real matrix

$$\begin{bmatrix} \dfrac{\lambda+\bar{\lambda}}{2} & \dfrac{\lambda-\bar{\lambda}}{2i} \\ -\dfrac{\lambda-\bar{\lambda}}{2i} & \dfrac{\lambda+\bar{\lambda}}{2} \end{bmatrix} \oplus \begin{bmatrix} \dfrac{\mu+\bar{\mu}}{2} & \dfrac{\mu-\bar{\mu}}{2i} \\ -\dfrac{\mu-\bar{\mu}}{2i} & \dfrac{\mu+\bar{\mu}}{2} \end{bmatrix}.$$

Next assume $W_1(A)$ is an x-symmetric triangle. Then A must have a real eigenvalue r and two complex conjugate eigenvalues λ and $\bar{\lambda}$. Since $t = \operatorname{tr}(A)$ is real, the remaining eigenvalue $s = t - 2\operatorname{Re}\lambda - r$ is real. Thus A is unitarily similar to

$$\begin{bmatrix} \dfrac{\lambda+\bar{\lambda}}{2} & \dfrac{\lambda-\bar{\lambda}}{2} \\ -\dfrac{\lambda-\bar{\lambda}}{2i} & \dfrac{\lambda+\bar{\lambda}}{2} \end{bmatrix} \oplus \operatorname{diag}(r, s).$$

If $W_1(A)$ is a line segment and x-symmetric, then either $W_1(A) \subseteq \mathbb{R}$ or $W_1(A) \subseteq p + i\mathbb{R}$, p real. The first case is simple, since then A is Hermitian. In the second case the x-symmetry implies that the vertex eigenvalues have the form

$$p \pm ir,$$

and the fact that $\operatorname{tr}(A) \in \mathbb{R}$ implies that the remaining two eigenvalues have the form

$$p \pm is.$$

Thus A is unitarily similar to

$$pI_4 + \left(\begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & s \\ -s & 0 \end{bmatrix} \right).$$

Finally, if $W_1(A)$ is a point and hence real, A is a real multiple of I_4. ∎

To summarize, we know the following:

$n = 2$: $W_1(A)$ is x-symmetric if and only if A is unitarily similar to a real matrix.

$n = 3$: There exist nonnormal A for which $W_1(A)$ is x-symmetric, and yet no nonzero multiple of A is unitarily similar to a real matrix.

$n = 3$: If A is normal and $W_1(A)$ is x-symmetric, then to within a real translation, followed by a rotation of A through an integral multiple of $\pi/2$, A is unitarily similar to a real matrix.

$n = 4$: There exist normal A for which $W_1(A)$ is x-symmetric and yet no translation and rotation of A is unitarily similar to a real matrix. Moreover, $W_2(A)$ is not x-symmetric.

$n = 4$: If A is normal, $W_1(A)$ is x-symmetric, and $\operatorname{tr}(A)$ is real, then A is unitarily similar to a real matrix.

$n = 5$: There exists a normal A with $W_1(A)$ x-symmetric, $\operatorname{tr}(A)$ real, and yet A not unitarily similar to a real matrix. Moreover $W_2(A)$ is not x-symmetric.

This last statement for $n = 5$ is confirmed by considering the matrix

$$A = \operatorname{diag}\left(i,\ -i, 1, \frac{i}{2}, \frac{1}{4} - \frac{i}{2}\right).$$

Clearly A cannot be unitarily similar to a real matrix, because the nonreal eigenvalues do not occur in complex conjugate pairs. Note that $W_2(A)$ is the convex hull of the numbers

$$0,\quad i+1,\quad \frac{3i}{2},\quad \frac{1}{4}+\frac{i}{2},\quad 1-i,\quad -\frac{i}{2},\quad \frac{1}{4}-\frac{3i}{2},\quad 1+\frac{i}{2},\quad \frac{5}{4}-\frac{i}{2},\quad \frac{1}{4}. \tag{58}$$

But it is easy to check that $\frac{5}{4} - i/2$ is strictly to the right of the remaining nine points and hence is a vertex. However, $\frac{5}{4} + i/2$ is not in the convex hull of the numbers (58), and so $W_2(A)$ is not x-symmetric. It is therefore interesting to note that Corollary 1 implies that for $n = 5$, if $W_1(A)$ and $W_2(A)$ are x-symmetric and $\operatorname{tr}(A)$ is real, then A is unitarily similar to a real matrix.

III. PROOFS

We begin with the proof of Theorem 1 and assume first that $W_k(A)$ is x-symmetric for $k = 1,\dots, n$. To prove that the characteristic polynomial of A is real it suffices to show that the complex eigenvalues of A occur in complex conjugate pairs.

Define an equivalence relation on the spectrum of A by assuming that two eigenvalues are equivalent if and only if they have equal imaginary parts. The resulting equivalence classes are disjoint subsets of the spectrum of A, each of which is contained in a unique horizontal line. Lines in the upper half plane will be denoted by $L_1, L_2,\dots, L_p$, and those in the lower half plane by $M_1, M_2,\dots, M_q$.

It is obvious from (21) that a real translation of A does not effect the x-symmetry of $W_k(A)$ or the possibility that the nonreal eigenvalues of A occur in complex conjugate pairs. Thus we can translate A as necessary through a real displacement so that $W_1(A)$ is entirely contained in the nonnegative right half plane and the imaginary axis is a support line for $W_1(A)$. We choose the notation so that the eigenvalues on L_s are subscripted according to decreasing real part:

$$L_s: \lambda_{s1}, \lambda_{s2},\dots, \lambda_{sm_s}, \tag{59}$$

where

$$\lambda_{sj} = a_{sj} + ib_s, \qquad j = 1,\dots, m_s, \tag{60}$$

and

$$a_{s1} \geqslant a_{s2} \geqslant \cdots \geqslant a_{sm_s} \geqslant 0, \tag{61}$$

$$b_s > 0. \tag{62}$$

Also we fix the notation so that

$$b_1 > b_2 > \cdots > b_p > 0. \tag{63}$$

Similarly, the eigenvalues on M_t are

$$M_t: \mu_{t1}, \mu_{t2},\dots, \mu_{tn_t}, \tag{64}$$

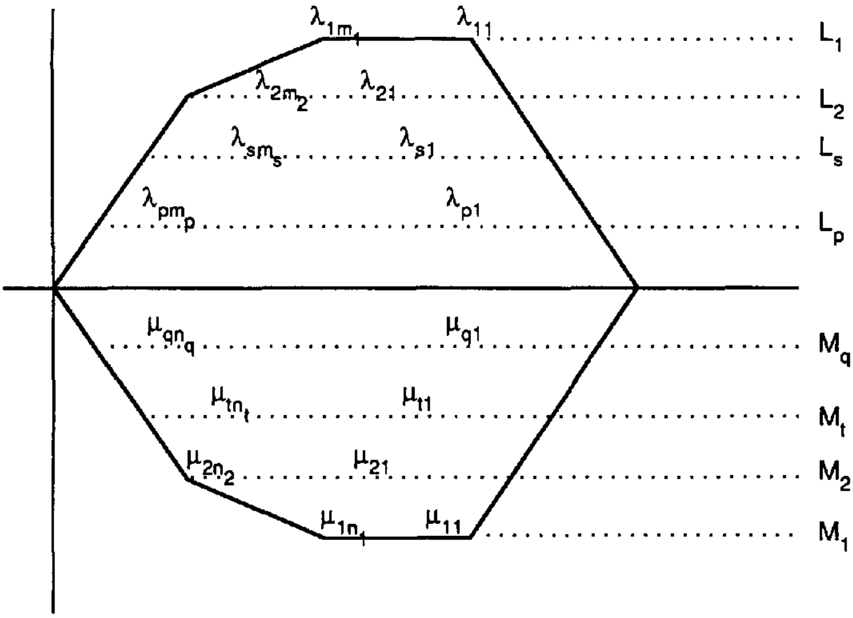

FIG. 1.

where

$$\mu_t = c_{tj} - id_t, \qquad j = 1,\dots,n_t, \tag{65}$$

and

$$c_{t1} \geqslant c_{t2} \geqslant \cdots \geqslant c_{tn_t} \geqslant 0, \tag{66}$$

$$d_t > 0. \tag{67}$$

Also,

$$d_1 > d_2 > \cdots > d_q > 0. \tag{68}$$

The situation is depicted in Figure 1.

Of course, it is possible that none of the eigenvalues of A are in the upper half plane. But then the x-symmetry of $W_1(A)$ implies A does not have eigenvalues in the lower half plane. Since $W_1(A)$ is a containment region for the spectrum of A, it follows that A is Hermitian and hence that the characteristic polynomial of A is real. We show first that the eigenvalues on L_1 and M_1 can be grouped into conjugate pairs and that $m_1 = n_1$. We then continue the argument to show that L_2 and M_2, L_3 and M_3, etc. are related in the same way until the set of lines in one of the half planes is exhausted, and hence that $p = q$.

Before continuing, we use the following fact about any convex polygon P in the plane. Suppose P is the convex hull of the points $z_1, z_2, \ldots, z_\tau$, with the notation so chosen that $z_1, \ldots, z_\nu$ are the highest among $z_1, z_2, \ldots, z_\tau$, i.e.,

$$\mathrm{Im}(z_1) = \cdots = \mathrm{Im}(z_\nu) > \mathrm{Im}(z_j), \qquad j = \nu + 1, \ldots, \tau. \tag{69}$$

In addition we choose the notation so that z_1 is farthest to the right among $z_1, \ldots, z_\nu$, i.e.,

$$\mathrm{Re}(z_1) \geqslant \mathrm{Re}(z_j), \qquad j = 1, \ldots, \nu. \tag{70}$$

It then follows that z_1, which is farthest to the right among the highest points $z_1, \ldots, z_\nu$, must be a vertex of P. Similarly, the point farthest to the right among the lowest points in the set $z_1, z_2, \ldots, z_\tau$ is a vertex of P. In case P is $W_k(A)$ we will denote by $\mathrm{NE}_k(A)$ the vertex of $W_k(A)$ defined by the conditions that describe z_1, i.e., the "northeast" vertex of $W_k(A)$. Similarly, $\mathrm{SE}_k(A)$ will denote the "southeast" vertex of $W_k(A)$, i.e., the point farthest to the right among the lowest sums of eigenvalues of A, taken k at a time [see (18)]. Note that when $W_k(A)$ is x-symmetric, then

$$\overline{\mathrm{NE}_k(A)} = \mathrm{SE}_k(A). \tag{71}$$

Since $\lambda_{11} = \mathrm{NE}_1(A)$ and $\mu_{11} = \mathrm{SE}_1(A)$, it follows from (71) that

$$\bar{\lambda}_{11} = \mu_{11}.$$

Suppose we have proved that

$$\bar{\lambda}_{ij} = \mu_{ij}, \qquad j = 1, \ldots, k-1 < m_1. \tag{72}$$

From (61) with $s = 1$ we see that

$$\mathrm{NE}_k(A) = \lambda_{11} + \lambda_{12} + \cdots + \lambda_{1k-1} + \lambda_{1k}. \tag{73}$$

From (66) with $t = 1$ we also conclude that

$$\mathrm{SE}_k(A) = \mu_{11} + \mu_{12} + \cdots + \mu_{1k-1} + \mu_{1k}, \tag{74}$$

From (71), (72), (73) and (74) it follows that

$$\bar{\lambda}_{ik} = \mu_{ik}. \tag{75}$$

The induction thus shows that

$$\bar{\lambda}_{ij} = \mu_{ij}, \qquad j = 1, \ldots, m_1. \tag{76}$$

Suppose that $m_1 < n_1$. Clearly

$$\mu_{11} + \cdots + \mu_{1m_1} + \mu_{1m_1+1} = \mathrm{SE}_{m_1+1}(A),$$

and hence by (71)

$$\bar{\mu}_{11} + \cdots + \bar{\mu}_{1m_1} + \bar{\mu}_{1m_1+1} = \mathrm{NE}_{m_1+1}(A). \tag{77}$$

But

$$\mathrm{NE}_{m_1+1}(A) = \lambda_{11} + \cdots + \lambda_{1m_1} + \lambda, \tag{78}$$

where λ is the eigenvalue of A farthest to the right on the first line strictly below L_1 (this line may be the real axis or one of the M_i). However, (76), (77), and (78) imply that $\lambda = \bar{\mu}_{1m_1+1}$. However, from (76),

$$\begin{aligned} \mathrm{Im}(\lambda) &= \mathrm{Im}(\bar{\mu}_{1m_1+1}) \\ &= d_1 \\ &= b_1, \end{aligned}$$

but since λ is strictly below L_1, $\mathrm{Im}(\lambda) < b_1$, a contradiction. Hence $m_1 = n_1$ and the eigenvalues on L_1 and M_1 match up in complex conjugate pairs.

Unless $p = q = 1$, in which case the proof is complete, there are two possibilities: either one of p or q is 1, or both $p > 1$ and $q > 1$. In the first case we can assume that $q = 1$ and $p > 1$, so that the set of lines in the lower half plane is exhausted while lines remain in the upper half plane. If we compute

$$\operatorname{tr}(A)=W_n(A), \tag{79}$$

it must be a number with positive imaginary part, because it is the sum of all the eigenvalues of A. However, the x-symmetry of $W_n(A)$ implies that (79) is real. Thus we cannot have $p>1$ and $q=1$, so that we can assume that lines remain below L_1 in the upper half plane and above M_1 in the lower half plane.

Suppose that we have proved that the eigenvalues on L_k and M_k match up in complex conjugate pairs for $k=1,\ldots,s$, i.e.,

$$\lambda_{kj}=\bar{\mu}_{kj}, \qquad j=1,\ldots,m_k=n_k, \tag{80}$$

holds for $k=1,\ldots,s$. Unless $s=p=q$, in which case the proof is complete, there are two possibilities: either $s=p$ or $s=q$, or there are more than s lines in each half plane. Suppose, for example, that $s=q$ but that $s<p$. Then, as before, (79) would be a number with a positive imaginary part, contradicting the x-symmetry of (79). Thus we can assume that both $s<p$ and $s<q$, and by replacing A with $\bar{A}$ if necessary, that $m_{s+1}\leqslant n_{s+1}$. Let

$$\sigma_s=\sum_{k=1}^{s}\sum_{j=1}^{m_k}\lambda_{kj},$$

$$\gamma_s=\sum_{k=1}^{s}\sum_{j=1}^{n_k}\mu_{kj},$$

$$p_s=\sum_{j=1}^{s}m_j,$$

and

$$q_s=\sum_{j=1}^{s}n_j.$$

From (80)

$$\bar{\sigma}_s=\gamma_s \tag{81}$$

and

$$p_s = q_s. \tag{82}$$

Consider the sum of $p_s + 1$ eigenvalues of A,

$$\sigma_s + \lambda_{s+1,1} \in W_{p_s+1}(A).$$

Obviously

$$\mathrm{NE}_{p_s+1}(A) = \sigma_s + \lambda_{s+1,1},$$

and hence is a vertex of $W_{p_s+1}(A)$. By the x-symmetry of $W_{p_s+1}(A)$

$$\begin{aligned}\bar{\sigma}_s + \bar{\lambda}_{s+1,1} &= \overline{\mathrm{NE}_{p_s+1}(A)} \\ &= \mathrm{SE}_{p_s+1}(A)\end{aligned}$$

and hence must be a vertex of $W_{p_s+1}(A)$. But clearly

$$\gamma_s + \mu_{s+1,1} = \mathrm{SE}_{p_s+1}(A),$$

and by (81)

$$\bar{\lambda}_{s+1,1} = \mu_{s+1,1}. \tag{83}$$

The argument now proceeds by a separate induction, similar to the one used to prove (76), to establish that

$$\bar{\lambda}_{s+1,j} = \mu_{s+1,j}, \qquad j = 1,\dots,m_{s+1},$$

and that

$$m_{s+1} = n_{s+1}.$$

Hence (80) holds for $k = s+1$. Thus (80) holds for $s = 1,\dots,\min\{p,q\}$. Again we can repeat the argument following (79) to conclude finally that $p = q$ and hence complete the necessity part of the proof of Theorem 1.

To prove the sufficiency is simpler. If the characteristic polynomial of A has real coefficients, then the nonreal eigenvalues of A occur in complex conjugate pairs. Thus conjugation acts as a permutation on the set of sums taken k at a time of the eigenvalues of A. Hence, since A is normal, by (18) we have

$$\begin{aligned}\overline{W_k(A)} &= \overline{P_k(A)}\\ &= P_k(A)\\ &= W_k(A). \end{aligned} \tag{84}$$

This completes the proof of Theorem 1.

We proceed to the proof of Corollary 1. Assume first that A is unitarily similar to a real matrix Then the nonreal eigenvalues of A occur in complex conjugate pairs, and the argument (84) shows that $W_k(A)$ is x-symmetric for $k=1,\dots,n$.

Conversely, suppose $W_k(A)$ is x-symmetric for $k=1,\dots,[n/2]$ and that $\operatorname{tr}(A)$ is real, i.e., $W_n(A)$ is x-symmetric. Then (19) implies that $W_k(A)$ is x-symmetric for $k=1,\dots,n$. By Theorem 1 we can assume that the nonreal eigenvalues of A are

$$a_k \pm ib_k, \qquad k=1,\dots,m, \tag{85}$$

and the real eigenvalues are

$$r_{2m+1},\dots,r_n. \tag{86}$$

The real normal matrix

$$\sum_{k=1}^{m} \oplus \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix} \oplus \operatorname{diag}(r_{2m+1},\dots,r_n)$$

has the same eigenvalues as A and hence is unitarily similar to A.

We prove Theorem 2. Assume that A is unitarily similar to a real normal matrix. By Corollary 1, $W_k(A)$ is an x-symmetric polygon for $k=1,\dots,[n/2]$ and $\operatorname{tr}(A)$ is real. Conversely, if $W_k(A)$ is an x-symmetric polygon for $k=1,\dots,[n/2]$ and $\operatorname{tr}(A)$ is real, then by (18), A is normal, and hence by Corollary 1, A is unitarily similar to a normal matrix.

We conclude with a few remarks and suggestions for additional work. Clearly it would be interesting to determine conditions on $W_k(A)$, $k=1,\dots,n$, which are equivalent to A being unitarily similar to a real matrix. There may be some interesting connections between this problem and those in [6] in which A is assumed nilpotent. The symmetry of $W_k(A)$ with respect to several lines may have some interesting implications.

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